Is there a black hole minimum mass?

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Applying the first and generalised second laws of thermodynamics for a realistic process of near critical black hole formation, we derive an entropy bound, which is identical to Bekenstein's one for radiation. Relying upon this bound, we derive an absolute minimum mass $\sim 0.04\sqrt{g_*}m_{\rm Pl}$, where g_* and $m_{\rm Pl}$ is the effective degrees of freedom for the initial temparature and the Planck mass, respectively. Since this minimum mass coincides with the lower bound on masses of which black holes can be regarded as classical against the Hawking evaporation, the thermodynamical argument will not prohibit the formation of the smallest classical black hole. For more general situations, we derive a minimum mass, which may depend on the initial value for entropy per particle. For primordial black holes, however, we show that this minimum mass can not be much greater than the Planck mass at any formation epoch of the Universe, as long as g_* is within a reasonable range. We also derive a size-independent upper bound on the entropy density of a stiff fluid in terms of the energy density.

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Choptuik [1] numerically studied a self-gravitating system of a massless scalar field and revealed that there appear critical phenomena at the threshold of black hole formation. If a generic one-parameter (λ) family of initial data sets are prepared, the black hole mass M resulting from the time evolution obeys a scaling law $M\propto|\lambda-\lambda^*|^\gamma$ for $\lambda\approx\lambda^*$, where λ^* is the critical value and $\gamma(>0)$ is the critical exponent. The consequence is that there are black holes of arbitrarily small masses in the limit $\lambda\to\lambda^*$. This work has been generalised to a system of a radiation fluid [2]. In the cosmological context, critical phenomena at the threshold of primordial black hole formation have been found [3] but subsequently Hawke and Stewart [4] reported that there is a minimum mass $\sim 2\times10^{-4}$ times the mass $M_{\rm H}$ contained within the Hubble horizon due to shock wave formation based on numerical simulations with high-resolution shock capturing scheme and suggested the link with kink instability [5]. Here we discuss another possibility of minimum masses based on black hole thermodynamics. Complementarily, the maximum masses of primordial black holes for different formation scenarios have been studied [6, 7]. We adopt the units in which $c=\hbar=k=1$ and denote $G=m_{\rm Pl}^{-2}$.

The final state of complete gravitational collapse is an outgoing flux and a Kerr black hole, for which the area A of the event horizon is given by $A=4\pi m_{\rm Pl}^{-4}\{[M+(M^2-(J/M)^2)^{1/2}]^2+(J/M)^2\}$, where J is the angular momentum of the hole. Then, neglecting the rotation, the Bekenstein-Hawking entropy $S_{\rm BH}$ is given by

$$S_{\rm BH}=\frac{m_{\rm Pl}^2}{4}A\simeq4\pi\frac{M^2}{m_{\rm Pl}^2}. \tag{1}$$

The effects of rotation may affect the discussion by a factor of two or so.

Chisholm [8] implicitly assumes that the region which is going to be a black hole is isolated from its environment, applies the generalised second law of thermodynamics for the collapse of a radiation fluid, and gets a lower bound on (primordial) black hole masses as $M\gtrsim m_{\rm Pl}^2/T_{\rm i}\sim g_*^{1/4}m_{\rm Pl}^2\epsilon_{\rm i}^{-1/4}$ and a lower bound on the mass fraction f of the primordial black hole M to the horizon mass $M_{\rm H}$ as $f\gtrsim\sqrt{g_*}T_{\rm i}/m_{\rm Pl}=(g_*\epsilon_{\rm i})^{1/4}/m_{\rm Pl}$, where g_*, T and ϵ are the effective degrees of freedom of relativistic particles, the temperature and the energy density, respectively, and the suffix "i" indicates it is for the initial value of the collapse.

Here we adopt more realistic assumption that the region to be a black hole is surrounded by the outer region with nonnegligible pressure p rather than isolated. This assumption will be appropriate not only for primordial black holes but also for near critical black hole formation in an asymptotically flat spacetime. This external pressure can exert a considerable amount of positive work on this region. Let R, E and S be the radius, energy and entropy of this region. Let us assume that self-gravity of this region is sufficiently weak at the initial moment for near critical collapse. The first law then becomes $d(\epsilon R^3)=-pd(R^3)$ for the adiabatic process, where p is the pressure. For a radiation fluid $p=\epsilon/3$, it follows that ϵ is proportional to R^{-4} and E is proportional to R^{-1}. This energy increase is due to the work term exerted by the external pressure. When the radius of the region is approximately equal to the gravitational radius, i.e., $M\simeq(R_{\rm i}/R_{\rm g})E_{\rm i}$, where $R_{\rm g}\simeq2M/m_{\rm Pl}^2$, a black hole forms. This yields

$$M^2\simeq\frac{m_{\rm Pl}^2}{2}E_{\rm i}R_{\rm i}. \tag{2}$$

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Applying the generalised second law $S_{\rm BH} \geq S_{\rm i}$, we get

$$2\pi E_{\rm i} R_{\rm i} \geq S_{\rm i}. \tag{3}$$

This is identical to the entropy bound that Bekenstein [9] invented from a different derivation. It should be emphasised that we can only have a trivial lower bound

$$M \geq m_{\rm Pl}\sqrt{\frac{S_{\rm i}}{4\pi}} \tag{4}$$

on black hole masses at this stage unlike Chisholm [8]'s argument based on the assumption of isolation.

Below we see that the right-hand side of Eq. (4) is also bounded from below through the entropy bound. Equation (3) implies that the radius of the region destined to be a black hole must be larger than $S/(2\pi E)$. Assuming that the region to be a black hole is approximately homogeneous, the radius of this region is bounded from below as

$$R_{\rm i} \gtrsim \frac{2}{3\pi T_{\rm i}}, \tag{5}$$

where we have used $s = 4\epsilon/(3T)$. Equation (5) might be seen along the line of Heisenberg's uncertainty relation, since T gives the characteristic momentum of photons. The entropy contained in this region is then bounded from below as

$$S_{\rm i} \gtrsim \frac{4\pi}{3} s_{\rm i} \left(\frac{s_{\rm i}}{2\pi\epsilon_{\rm i}}\right)^3 = \frac{64}{3645} g_*, \tag{6}$$

where we have used $\epsilon = (\pi^2/30) g_* T^4$. Then, the generalised second law yields a minimum mass

$$M \gtrsim \frac{4}{27}\sqrt{\frac{g_*}{5\pi}} m_{\rm Pl}, \tag{7}$$

and therefore a lower bound

$$f \gtrsim \frac{16\pi}{405} g_* \left(\frac{T}{m_{\rm Pl}}\right)^2. \tag{8}$$

It should be noted that the above lower bound is very different from Chisholm [8]'s estimates. If g_* is order 1000, this minimum mass is order the Planck mass. This argument implies that the Planck mass naturally arises as a black hole minimum mass even without explicit form of quantum gravity. This also implies that the generalised second law naturally requires quantum gravity and the former should be justified by the latter. If g_* is much larger than order 1000, such as in Hagedorn-type scenarios [10], the minimum mass may become much larger than the Planck mass.

In the process of gravitational collapse to a black hole, there is a possibility that a heat flux may be induced by radiation transfer and/or shocks. In such a case, the system is no longer adiabatic and cannot be described by a perfect fluid alone. If the heat flux carries a considerable amount of heat away from the region within the time scale of collapse, the entropy in the collapsing region could decrease and the resultant black hole mass could be much smaller than the estimated value here. It will be interesting to investigate this possibility further based on specific processes of heat flux generation.

A black hole emits black body radiation due to quantum particle creation and its temperature is given by $T_{\rm H} = m_{\rm Pl}^2/(8\pi M)$ [11, 12]. The energy loss rate due to the Hawking radiation for a black hole of mass M is estimated as [13]

$$\frac{dM}{dt} = -\frac{g_*\Gamma}{15360\pi}\frac{m_{\rm Pl}^4}{M^2}, \tag{9}$$

where Γ is a positive constant of order unity. From this, the evaporation time $t_{\rm ev}$ is calculated as

$$t_{\rm ev} = \frac{5120\pi}{g_*\Gamma}\left(\frac{M}{m_{\rm Pl}}\right)^3 m_{\rm Pl}^{-1}. \tag{10}$$

Since the classical dynamical time of the black hole is estimated by the light crossing time of the horizon radius and calculated as $t_{\rm dyn} = 2(M/m_{\rm Pl})m_{\rm Pl}^{-1}$, the black hole can be regarded as classical if and only if $t_{\rm ev} \gtrsim t_{\rm dyn}$ or

$$M \gtrsim \frac{1}{16}\sqrt{\frac{g_*\Gamma}{10\pi}} m_{\rm Pl}. \tag{11}$$

The right-hand side gives another lower bound for classical black holes and this approximately agrees with the minimum mass given by Eq. (7) from the thermodynamical second law argument up to the factor of $\sqrt{g_*}$. In other words, the thermodynamical minimum mass argument will not prohibit the formation of the smallest classical black holes.

The thermodynamical minimum mass argument can be easily extended to a general perfect fluid with the equation of state $p = \alpha\epsilon$, where α is a constant. We assume $0 \leq \alpha \leq 1$ to respect causality and thermodynamical stability. The first law then yields $M \simeq (R_{\rm i}/R_g)^{3\alpha} E_{\rm i}$ and we have

$$M = \left(\frac{m_{\rm Pl}^2 R_{\rm i}}{2}\right)^{3\alpha/(1+3\alpha)} E_{\rm i}^{1/(1+3\alpha)}. \tag{12}$$

Applying the generalised second law $S_{\rm BH} \geq S_{\rm i}$, we get

$$4\pi m_{\rm Pl}^{-2(1-3\alpha)/(1+3\alpha)} \left(\frac{R_{\rm i}}{2}\right)^{6\alpha/(1+3\alpha)} E_{\rm i}^{2/(1+3\alpha)} \geq S_{\rm i}. \tag{13}$$

This is an entropy bound for the general case. For $0 \leq \alpha < 1$, this bound can be regarded as providing the minimum radius of the region to be a black hole:

$$R_{\rm i} \gtrsim m_{\rm Pl}^{2(1-3\alpha)/[3(1-\alpha)]} \left(\frac{s_{\rm i}^{1+3\alpha}}{\epsilon_{\rm i}^2}\right)^{1/[3(1-\alpha)]}, \tag{14}$$

where and hereafter we omit numerical factors. The entropy of the collapsing region is bounded from below as

$$S_{\rm i} \simeq \frac{4\pi}{3} s_{\rm i} R_{\rm i}^3 \gtrsim m_{\rm Pl}^{2(1-3\alpha)/(1-\alpha)} \left(\frac{s_{\rm i}}{\epsilon_{\rm i}^{1/(1+\alpha)}} \right)^{2(1+\alpha)/(1-\alpha)} . \tag{15}$$

Then, the generalised second law yields

$$M \gtrsim m_{\rm Pl}^{(1-3\alpha)/(1-\alpha)} \left(\frac{s_{\rm i}}{\epsilon_{\rm i}^{1/(1+\alpha)}} \right)^{(1+\alpha)/(1-\alpha)} \cdot m_{\rm Pl}. \tag{16}$$

Since $\epsilon \propto V^{-(1+\alpha)}$, where V is the volume,

$$n \equiv g_* [\epsilon/(g_* m_{\rm Pl}^4)]^{1/(1+\alpha)} m_{\rm Pl}^3, \tag{17}$$

has the dimension of energy cubed, is proportional to the degrees of freedom g_* and $N \equiv nV$ is constant and nondimensional. Therefore, we can identify n with the conserved number density of particles. Using this fact, we can rewrite Eqs. (15) and (16), respectively, as

$$S_{\rm i} \gtrsim g_*^{2\alpha/(1-\alpha)} \left(\frac{S}{N} \right)^{2(1+\alpha)/(1-\alpha)}, \tag{18}$$

$$M \gtrsim g_*^{\alpha/(1-\alpha)} \left(\frac{S}{N} \right)^{(1+\alpha)/(1-\alpha)} \cdot m_{\rm Pl}, \tag{19}$$

where N is the conserved number of particles, such as photons. This means that, in general, the minimum mass can depend on the entropy per particle. If $\epsilon = \epsilon(T)$, $dE = TdS - pdV$ and the energy equation $(\partial E/\partial V)_T = T(\partial p/\partial T)_V - p$ yield $\epsilon = \tilde{\beta} T^{(1+\alpha)/\alpha}$ and $s = (1+\alpha)\tilde{\beta} T^{1/\alpha}$, where $\tilde{\beta}$ is a constant. We can write $\tilde{\beta} = \beta C^{(3\alpha-1)/\alpha}$, where β is a nondimensional constant and C is a constant which has the dimension of energy. Equations (15) and (16) then become respectively

$$S_{\rm i} \gtrsim \beta^{2\alpha/(1-\alpha)} (1+\alpha)^{2(1+\alpha)/(1-\alpha)} \left(\frac{m_{\rm Pl}}{C} \right)^{2(1-3\alpha)/(1-\alpha)} \tag{20}$$

$$M \gtrsim \beta^{\alpha/(1-\alpha)} (1+\alpha)^{(1+\alpha)/(1-\alpha)} \left(\frac{m_{\rm Pl}}{C} \right)^{(1-3\alpha)/(1-\alpha)} m_{\rm Pl} \tag{21}$$

Therefore, if $C \ll m_{\rm Pl}$ for $0 < \alpha < 1/3$ or $C \gg m_{\rm Pl}$ for $1/3 < \alpha < 1$, the black hole minimum mass gets much larger than the Planck mass.

It should be noted that we need a special treatment for a stiff fluid $\alpha = 1$. A massless scalar field can be also regarded as a stiff fluid if and only if its gradient is timelike. In this case, Eq. (13) does not yield a lower bound on $R_{\rm i}$. Instead, we have a size-independent bound on the entropy density in terms of the energy density as

$$s \lesssim m_{\rm Pl} \sqrt{\epsilon}. \tag{22}$$

For this case, we have no black hole minimum mass because there is no lower bound on the entropy contained within the region to be a black hole.

For an interesting application, let us consider primordial black hole formation. Because of the second law, the ratio S/N is a nondecreasing function of time. We can assume that matter fields move together due to strong coupling. In this case, ϵ and α in the above discussion can be regarded as those for the dominant component in energy. Since we can neglect the effects of co-existing epochs, this ratio before the radiation-dominated era is equal to or smaller than the value at the radiation-dominated era, which is order unity. This means that the minimum mass before the radiation-dominated era is not much greater than $m_{\rm Pl}$ as long as g_* is within a reasonable range.

After the matter-radiation equality, nonrelativistic matter fields, such as cold dark matter and baryonic component, begins to dominate the Universe in energy but most of the entropy is still held in background thermal radiation. Gravitational collapse in this regime proceeds as the collapse of dark matter and the concentration of radiation due to the gravitational field will be negligible. This means that we can treat this process of black hole formation as that with $\alpha = 0$ and the ratio s/ϵ becomes effectively smaller than order unity because of the escape of entropy from the collapsing region. This means that the minimum mass of black holes becomes smaller than the Planck mass, although the application of this argument to black holes much smaller than the Planck mass is limited.

These considerations lead to the conclusion that since the total entropy contained within the Universe does not decrease in time, which follows from the second law of thermodynamics, the thermodynamic minimum mass of primordial black holes is not much greater than $m_{\rm Pl}$ at any formation epoch of the Universe, unless g_* is extremely large. It is expected that the modification of observational constraints due to critical behaviour [14, 15, 16] will not be so sensitive to the existence of minimum mass discussed here.

After this paper was received, Ref. [8] has been significantly revised and published as Ref. [17]. The lower bound obtained in [17] is very similar to Eq. (7) in the current paper. However, they are very different in derivation and therefore assumptions. In Ref. [17], the lower bound directly comes from the entropy of radiation immediately before the black hole formation. Hence, g_* is regarded as $g_*(T_{\rm f})$ for the temperature $T_{\rm f}$ at the time of formation. The discussion there crucially relies upon the assumption that the matter field is a radiation fluid at the formation. Moreover, since self-gravity is of course very strong and the system is highly dynamical immediately before the horizon formation, the assumption that entropy only comes from radiation yet needs to be checked. In fact, Penrose [18] introduced the concept that entropy can reside in a strong gravitational field and this is called gravitational entropy. See also [19] and references therein. The contribution from gravitational entropy could in principle significantly change the total entropy immediately before the black hole formation. In the current paper, in contrast, since the derivation is given in terms of the initial configuration well before the black hole formation, g_* can be regarded as $g_*(T_{\rm i})$ for the initial temperature $T_{\rm i}$ and the assumption of weak self-gravity is justifiable in a self-consistent manner. Even if we take uncertainty in matter field models at very high energy scale and/or strong self-gravity at horizon formation into account, the thermodynamical argument developed in this paper will be modified only in terms of small correction terms.

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